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1991 J. Phys.: Condens. Matter 3 6509

(http://iopscience.iop.org/0953-8984/3/33/025)

J. Phys.: Condens. Matter 3 (1991) 6509–6521. Printed in the UK

Time-dependent and steady-state statistics of photons at non-zero chemical potential

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Received 12 July 1990, in final form 14 March 1991

Abstract. The paper analyses some time-dependent and steady-state properties of coupled photons and electrons in two-level systems, by taking into consideration the radiative transitions and photon dissipation. The main assumption is the slaving principle for adiabatic elimination of the electrons. A master equation is derived for the photon number. The stability of the photon population is studied and the laser threshold concentration of the upper-level electrons determined. A first-order phase transition that occurs at threshold is established explicitly, and it is shown that the transition requires the neglect of the spontaneous emission and of the ambient influence. The steady-state photon distribution is not Poissonian and is characterized by large fluctuations. Fluctuations become negligible when the stimulated emission is zero. The steady-state distribution can obey the Bose–Einstein law with a non-zero chemical potential. The equation of state of the photon population is derived. When dissipation is weak and dilution increases, the equation of state is similar to that of a degenerate Bose gas in equilibrium.

1. Introduction

A well known result shows that black-body radiation has a zero chemical potential. In this case Planck's law gives the photon distribution. However, the situation is different when photons are interacting with an electronic system in quasi-equilibrium. So, in the case of photons coupled with a two-level electron system, Landsberg (1981) proved that the quasi-Fermi levels of the electron distribution can superimpose themselves on the radiation so as to turn it into a non-equilibrium steady-state Planck-type distribution which has acquired what appears to be a non-zero chemical potential. There is a fair amount of literature on this non-equilibrium steady-state distribution (Chel'tzov 1971, Haught 1984, Henry 1980, Ruppel and Wurfel 1980, Wurfel and Ruppel 1980, 1981, 1985, Wurfel 1982).

The non-equilibrium kinetics of coupled photons and electrons were studied in more detail by Schöll and Landsberg (1983). These workers approached the problem for the case of a two-level system with strongly interacting atoms, e.g. a semiconductor in which the electrons are not localized at individual atoms. Their object was to develop a theory that covers the whole range from thermal equilibrium to the laser regime. With this purpose, Schöll and Landsberg proposed a model which took into consideration

(i) the radiative electronic transitions,
(ii) the non-radiative electronic excitations,

0953-8984/91/336509 + 13 $03.50

(iii) the non-radiative electronic recombinations and
(iv) the photon dissipation.

By using the model they established the coupled rate equations for the average numbers of photons and upper-level electrons, $\bar{N}$ and $\bar{n}$, respectively:

$$\dot{\bar{N}} = u^{\text{rad}} - u^{\text{diss}} \tag{1}$$

$$\dot{\bar{n}} = -u^{\text{rad}} - u^{\text{rec}} + u^{\text{exc}} \tag{2}$$

where u^{rad}, u^{rec}, u^{exc} and u^{diss} denote the transition rates of processes (1)–(4). Schöll and Landsberg were not able to obtain analytically the general time-dependent solution of the non-linear coupled equations (1) and (2). However, they assumed a separation of time scales that allows one to calculate the evolution of the photon distribution in time. Indeed, the electrons can often be considered as fast variables compared with the photons since for weak photon dissipation the photons relax slowly to the steady state (see, e.g., Schöll and Landsberg (1983) and references therein). Consequently, Schöll and Landsberg first considered the electrons as being in a steady state ($\dot{\bar{n}} = 0$). Then, they obtained from equation (2) a dependence $\bar{n} = \bar{n}(\bar{N})$ by determining u^{rad}, u^{rec} and u^{exc}. This dependence was used in (1) in order to make this equation self-consistent in $\bar{N}$. As they observed, because of the dependence $\bar{n} = \bar{n}(\bar{N})$ the electron number $\bar{n}$ is still varying but only as a function of $\bar{N}(t)$, which varies slowly. They called this time variation of $\bar{n}$ a pseudosteady state. Even under these simplifications, obtaining the time variation of $\bar{N}$ analytically is a difficult task. Schöll and Landsberg (1983) give the results obtained for two special cases and the so-called full steady state ($\dot{\bar{n}} = 0$; $\dot{\bar{N}} = 0$).

In this paper we analyse the same system of coupled photons and electrons which Schöll and Landsberg (1983) studied. One of the main assumptions that we accepted is again the hypothesis of the pseudosteady state of the upper-level electrons. However, our analysis is performed by neglecting the non-radiative processes. The above restriction makes it possible to develop and use a probabilistic technique based on the master equation method. Consequently, we obtain new results, which cannot be derived by using deterministic techniques, such as the method of the rate equations.

2. Photons coupled with a two-level system of electrons

Following Schöll and Landsberg (1983) and Landsberg (1986) we consider a pair of electronic energy levels E_I and E_J, where $E_G = E_I - E_J \equiv h\nu > 0$. Let the number of available quantum states at these levels be N_I and N_J. We denote the number of electrons in the upper and lower levels by n and m, respectively. Assume the total number of electrons

$$N_e \equiv n + m \tag{3}$$

to be conserved. Physically meaningful values must clearly satisfy

$$0 \leqslant n \leqslant N_e < N_I + N_J. \tag{4}$$

Electron transitions between the two levels determine the absorption and emission of photons of energy $h\nu$. The photon number is N. We denote the three types of particles by $\{n\}$, $\{m\}$ and $\{N_\nu\}$, respectively.

Neglecting the non-radiative excitations and combinations, the relative population of the levels I and J may change because of the following processes:

(i) radiative transitions (absorption and spontaneous and stimulated emission) with transition probabilities per unit time, B_a, B_e^{sp} and B_e^{st}, respectively (for convenience later we still distinguish the three coefficients, although they are in fact identical (Landsberg 1986)).

(ii) photon dissipation processes (e.g. absorption, scattering and cavity loss).

Scattering implies the existence of

(a) photons of wavelength $\nu' \neq \nu$ and

(b) material particles other than electrons $\{n\}$ and $\{m\}$.

We denote by $\{N_{\nu'}\}$ and $\{p\}$ the types of particles from (a) and (b) above. Schöll and Landsberg (1983) called the whole set of $\{N_{\nu'}\}$ and $\{p\}$ particles 'the ambient'.

The processes (i) and (ii) may be formally described by

$$\{m\} + \{N_\nu\} \underset{B_e}{\overset{B_a}{\rightleftarrows}} \{n\} \tag{5}$$

$$\{p\} + [N_\nu\} \underset{k}{\overset{k}{\rightleftarrows}} \{p\} + \{N_{\nu'}\} \tag{6}$$

where k is the transition probability per unit time which characterizes the photon dissipation process.

The ensemble of variables determining the macroscopic state of the system that we study is (N, n, m). The probability that at time t the system be in a state (N, n, m) will be denoted $P(N, n, m, t)$. We denote by $w(N', n', m'; N, n, m)$ the transition probability from state (N', n', m') in state (N, n, m).

In the case of the electronic transition between level J and level I (absorption of a photon) we have

$$w_a(N+1, n-1, m+1; N, n, m) = B_a(N+1)(m+1)[N_I - (n-1)]. \tag{7}$$

Two different situations occur in the case of the electronic transition from level I to level J (emission of a photon). For stimulated emission we may write

$$w_e^{st}(N-1, n+1, m-1; N, n, m) = B_e^{st}(N-1)(n+1)[N_J - (m-1)] \tag{8}$$

and for spontaneous emission we have

$$w_e^{sp}(N-1, n+1, m-1; N, n, m) = B_e^{sp}(n+1)[N_J - (m-1)]. \tag{9}$$

Schöll and Landsberg (1983) took into account the photon loss $\dot{\bar{N}} = -k\bar{N}$ in order to overcome the problem of an infinite number of photons at the laser threshold. We use the same hypothesis for the photon dissipation transition probabilities;

$$w_{diss}(N+1, n, m; N, n, m) = k(N+1). \tag{10}$$

Schöll and Landsberg noted that at thermal equilibrium we need $\dot{\bar{N}} = 0$. Consequently, a new term of the form $\dot{\bar{N}} = kN_0$ must be added. Here N_0 is the number of photons in thermal equilibrium and kN_0 represents a radiation gain from the ambient. Again, we use their assumption and obtain:

$$w_g(N-1, n, m; N, n, m) = W_g(N, n, m; N+1, n, m) = kN_0. \tag{11}$$

Now, the master equation of the processes (i) and (ii) may be written

$$\begin{aligned}
\mathrm{d}P(N,n,m,t)/\mathrm{d}t = {} & w_{\mathrm{a}}(N+1,n-1,m+1;N,n,m)P(N+1,n-1,m+1,t) \\
& - w_{\mathrm{a}}(N,n,m;N-1,n+1,m-1)P(N,n,m,t) \\
& + w_{\mathrm{e}}^{\mathrm{st}}(N-1,n+1,m-1;N,n,m)P(N-1,n+1,m-1,t) \\
& - w_{\mathrm{e}}^{\mathrm{st}}(N,n,m;N+1,n-1,m+1)P(N,n,m,t) \\
& + w_{\mathrm{e}}^{\mathrm{sp}}(N-1,n+1,m-1;N,n,m)P(N-1,n+1,m-1,t) \\
& - w_{\mathrm{e}}^{\mathrm{sp}}(N,n,m;N+1,n-1,m+1)P(N,n,m,t) \\
& + w_{\mathrm{diss}}(N+1,n,m;N,n,m)P(N+1,n,m,t) \\
& - w_{\mathrm{diss}}(N,n,m;N-1,n,m)P(N,n,m,t) \\
& + w_{\mathrm{g}}(N-1,n,m;N,n,m)P(N-1,n,m,t) \\
& - w_{\mathrm{g}}(N,n,m;N+1,n,m)P(N,n,m,t). \qquad (12)
\end{aligned}$$

The probability $P(N,n,m,t)$ depends on only three independent variables, because n and m are related through (3). Even in this case solving the equation (12) is a difficult task. Considerable simplification occurs if we assume that the electrons are in a steady state. In this case, n is a constant and $P(N,n,m,t)$ depends only on N and t. However, N and n are coupled variables and the time evolution of N makes impossible a real steady state for n. As we pointed out in section 1, the compromise of Schöll and Landsberg (1983) was to accept the so-called pseudosteady state of $\bar{n}$. Their assumption is a direct consequence of the slaving principle for adiabatic elimination of the electrons (Haken 1975). The reason is that the relaxation time of photons is usually much greater than that of electrons (Haken 1970, p 249). Thus the electrons can follow the 'orders' of the photons adiabatically and n can be eliminated without increasing the degree of the time derivatives. The long lifetime of the photons allows the electrons to become 'slaves'. The pseudosteady state of $\bar{n}$ approximates the true trajectory $\bar{N}(\bar{n}(t),t)$ by the appropriate branch of the null isocline $\dot{\bar{n}}=0$, i.e. by $\bar{N}(\dot{\bar{n}}=0,t)$ (Schöll and Landsberg 1983, p 1202). A better approximation occurs around the full steady state ($\dot{n}=\dot{N}=0$) as the phase diagrams in figures 1, 2 and 5 in the work of Schöll and Landsberg (1983) show. We may conclude that proper application of the slaving principle occurs in the case of the weak dissipation processes which may attain the full steady state.

By using the slaving principle and equations (7)–(11), we obtain a new form of the master equation (12):

$$\begin{aligned}
\mathrm{d}P(N,t)/\mathrm{d}t = {} & A_1(N+1)P(N+1,t) - [A_2+(A_1+A_3)N]P(N,t) \\
& + [A_2+(N-1)A_3]P(N-1,t) \qquad (13)
\end{aligned}$$

where

$$A_1 \equiv B_{\mathrm{a}}(N_{\mathrm{e}}-n)(N_I-n)+k \equiv B_{\mathrm{a}}\beta + k \qquad (14)$$

$$A_2 \equiv B_{\mathrm{e}}^{\mathrm{sp}} n(N_J - N_{\mathrm{e}} + n) + kN_0 \equiv B_{\mathrm{e}}^{\mathrm{sp}}\alpha + kN_0 \qquad (15)$$

$$A_3 \equiv B_{\mathrm{e}}^{\mathrm{st}} n(N_J - N_{\mathrm{e}} + n) \equiv B_{\mathrm{e}}^{\mathrm{st}}\alpha. \qquad (16)$$

Equation (13) is a finite-difference equation with linear coefficients. The finite-difference property is a result of the discreteness of the stochastic variable—the number of photons—while the linearity of the coefficients is a consequence of the one-particle character of the relations (5) and (6).

3. Time-dependent photon statistics

A study of equation (13) is performed most conveniently in the generating-function representation. To this end we define the generating function $F(s, t)$ by (Bharucha-Reid 1960, p 18)

$$F(s, t) \equiv \sum_{N=0}^{\infty} s^N P(N, t) \tag{17}$$

where $|s| \leqslant 1$ in order to assure convergence. By using (17), equation (13) takes the following form:

$$\partial F/\partial t = [A_1 - (A_1 + A_3)s + A_3 s^2]\,\partial F/\partial s + A_2(s-1)F. \tag{18}$$

This is a first-order equation with partial derivatives. It will be solved by taking into account that firstly $P(N, t)$ must be normalized and secondly at time $t = 0$ the number of the photons $\{N_p\}$ is known, say $N(0)$. The first condition means that

$$F(1, t) = \sum_{N=0}^{\infty} P(N, t) = 1 \tag{19}$$

while the second implies that

$$P(N, 0) = \begin{cases} 1 \\ 0 \end{cases} \qquad \text{for} \begin{cases} N = N(0) \\ N \neq N(0). \end{cases} \tag{20}$$

Therefore

$$F(s, 0) = s^{N(0)}. \tag{21}$$

By using (21) we obtain the solution of equation (18) (see the appendix):

$$\begin{aligned} F(s, t) = {} & |\{A_1(1-s) - (A_1 - A_3 s)\exp[-(A_3 - A_1)t]\}/\{A_3(1-s) \\ & - (A_1 - A_3 s)\exp[-(A_3 - A_1)t]\}|^{N(0)}\,|\{(A_1 - A_3)(A_1 - A_3 s) \\ & \times \exp[-(A_3 - A_1)t]\}/\{A_3[-A_1 + (A_1 + A_3)s - A_3 s^2] \\ & + (A_1^2 - 2A_1 A_3 s + A_3^2 s^2)\exp[-(A_3 - A_1)t]\}|^{A_2/A_3}. \end{aligned} \tag{22}$$

Simple calculation shows that $F(s, t)$ verifies (19).

The generating function can be used directly to determine the stochastic function $P(N, t)$ (Bharucha-Reid 1960, p. 442) by

$$P(N, t) = F^{(N)}(0, t)/N! \qquad N = 1, 2, \ldots. \tag{23}$$

This rather tedious computational task is not carried out here. Also, the generating function can be used to determine the mean value $\bar{N}$ of the photon number (Bharucha-Reid 1960, p 440):

$$\bar{N} = (\partial F/\partial s)|_{s=1}. \tag{24}$$

From (22) and (24) we obtain

$$\bar{N}(t) = [N(0) - A_2/(A_1 - A_3)]\exp[(A_3 - A_1)t] + A_2/(A_1 - A_3). \tag{25}$$

The time variation of the mean number of photons mainly depends on the sign of the

difference $A_3 - A_1$. From (14) and (16) we see that $A_3 - A_1$ is the difference between the intensities of the stimulated emission and the absorption plus dissipation processes, respectively.

Note that $A_3 - A_1$ does not depend on spontaneous emission. The photon population is stable when the absorption plus dissipation process is more intensive than the stimulated emission process ($A_3 - A_1 < 0$). In this case, for any initial value $N(0)$ (which includes possible fluctuations) the mean photon number has a monotonically increasing or decreasing time variation between $N(0)$ and $A_2(A_1 - A_3)^{-1}$ with a relaxation time $(A_1 - A_3)^{-1}$. The photon population is not stable when the stimulated emission is stronger than the absorption plus dissipation process ($A_3 - A_1 > 0$). In this case the mean photon number increases in time indefinitely. When the two processes have the same intensity ($A_3 - A_1 = 0$), the photon population is at the laser threshold (Schöll and Landsberg 1983) and the photon mean number diverges. Note that in this case the elaxation time is infinite and the photon population becomes unstable without oscillations in time. Consequently, the well known analogy to a phase transition applies (see, e.g., Haken 1975). This analogy has been studied from different viewpoints by many workers (see, e.g., DeGiorgio and Scully 1970, Graham and Haken 1970).

Let us briefly analyse what implies $A_3 - A_1 \leqslant 0$. From (14) and (16) we obtain the following condition:

$$(B_c^{st} - B_a)n^2 + [B_c^{st}(N_J - N_e) + B_a(N_I + N_e)]n - B_a N_e N_I - k \leqslant 0. \tag{26}$$

However $B_c^{st} = B_a$ (Schöll and Landsberg 1983, Landsberg 1986). In this case from (26) we obtain a threshold condition for the excited electron concentration:

$$n \leqslant N_e N_I/(N_I + N_J) + k/[B_c^{st}(N_I + N_J)] \equiv n_{th} + n_k. \tag{27}$$

A similar result was derived by the previously quoted workers. When $n = n_{th} + n_k$ we obtain the threshold condition for laser action. As we can see, the laser threshold condition applies also in the case of a non-null dissipation although the energy of the photon ensemble is not conserved. Consequently, there exists a fundamental difference between the laser and a thermodynamic system showing a phase transition. This difference was pointed out also by other workers (see, e.g., Haken 1975, p 81).

Equation (25) as well as all the results obtained by using the rate equation method (Schöll and Landsberg 1983, Landsberg 1986) show that the average value of the photon number cannot vanish. However, what is the probability that $N = 0$? This can be determined from (Bharucha-Reid 1960, p 442)

$$P(0, t) = F(0, t). \tag{28}$$

By using (22) and (28) we find that

$$P(0,t) = |A_1\{1 - \exp[-(A_3 - A_1)t]\}/\{A_3 - A_1 \exp[-(A_3 - A_1)t]\}|^{N(0)} \times |(A_1 - A_3)\exp[-(A_3 - A_1)t]/\{A_1 \exp[-(A_3 - A_1)t] - A_3\}|^{A_2/A_3}. \tag{29}$$

The probability is non-null for both stable and unstable photon ensembles ($A_1 - A_3 \gtrless 0$). This means that the photon population may disappear as a result of fluctuations. However, note that $N = 0$ is not an absorbing state as spontaneous emission exists ($A_2 \neq 0$). Indeed, if we make $N(0) = 0$ in (29) we see that $P(0, t) \neq 1$ for $t > 0$. The photon population reappears. The probability $P(0, t)$ vanishes only for $A_3 = A_1$, i.e. at the laser threshold, when compulsorily $N \neq 0$. In the case of unstable photon

populations ($A_1 - A_3 > 0$), when $\bar{N}$ increases in time, the probability $P(0, t)$ decreases in time towards zero. When a stable population is considered ($A_1 - A_3 < 0$) the value of $P(0, t)$ has the asymptotic limit:

$$P(0, t \to \infty) = |1 - A_3/A_1|^{A_2/A_3}. \tag{30}$$

As we can see, the photon population can disappear even if the steady state is attained. Note that at steady state the probability $P(0, t \to \infty)$ does not depend on fluctuations (i.e. on $N(0)$).

New information concerning the time properties of the photonic ensemble can be obtained by computing the variance $D^2(N)$. This is a measure of the photon number fluctuations around its mean value $\bar{N}$. The variance can be obtained by (Bharucha-Reid 1960, p 440)

$$D^2(N(t)) = |\partial^2 F/\partial s^2 + \partial F/\partial s - (\partial F/\partial s)^2|_{s=1}. \tag{31}$$

By using (22), (25) and (31) we find that

$$D^2 = \bar{N}[1 + (A_3/A_2)\bar{N}] - N(0)[1 + (A_3/A_2)N(0)] \exp[2(A_3 - A_1)t] \qquad t > 0. \tag{32}$$

We see that generally $D^2 \neq \bar{N}$. Consequently, the distribution $P(N, t)$ is not Poissonian and the fluctuations are not negligible compared with the average values (Nicolis and Prigogine 1977, p 241). Also, $D^2(N(0)) = 0$. This is not surprising because we accepted the values $N(0)$ as being known.

Let us briefly analyse the particular case when $N(0) = 0$, i.e. when the photon population disappeared. As previously has been proved, this situation is possible whatever the difference between the intensity of the stimulated emission process and the intensity of the absorption plus dissipation process ($A_1 - A_3 \gtrless 0$). From (32) we see that $D^2/\bar{N}^2 = 1/\bar{N} + A_3/A_2$. This value is finite and non-null when $\bar{N} \to \infty$, i.e. near the laser threshold ($A_1 = A_3$) or, in other words, near the phase transition. We conclude that in this case the scales of macroscopic averages and fluctuations are not clearly separated. This means that the average value $\bar{N}$ may not be representative of the system (Nicolis and Prigogine 1977, p 228). There is a breakdown in the laws of large numbers, i.e. the 'sampling' procedure allowed by the laws of large numbers and consisting of identifying the arithmetic mean with the stochastic average is not entirely meaningful.

Important non-equilibrium features of the photon population can be demonstrated by studying the time variation of the ratio $D^2/\bar{N}$. First, let us analyse the asymptotic stability of the photon population fluctuations. In the beginning we considered the absorption plus dissipation process to be more intensive than stimulated emission ($A_3 - A_1 < 0$). Two subcases may be distinguished. When the initial photon population is $N(0) > A_2(A_1 - A_3)^{-1}$ the photon mean number decreases monotonically in time (see equation (25)). In this case, from (32) we observe that $D^2/\bar{N}(t_2 \to \infty) < D^2/\bar{N}(t_1 > 0)$. The fluctuations of the photon population decrease asymptotically. When the initial photon population is such that $N(0) < A_2(A_1 - A_3)^{-1}$, by using equation (25) we see that $\bar{N}(t)$ increases in time. Now, equation (32) shows that $D^2/\bar{N}(t_2 \to \infty) > D^2/\bar{N}(t_1 > 0)$, i.e. the fluctuations increase asymptotically. From the two subcases we observe that in the asymptotic limit $t \to \infty$ both the average value $\bar{N}(t)$ of the photon population and its relative fluctuation have the same time variation, but both the increase and the decrease in $\bar{N}(t)$ are limited by the value $A_2(A_1 - A_3)^{-1}$. Consequently, the fluctuations are also limited. In the asymptotic limit $t \to \infty$ the photon population is stable at fluctuations.

If the stimulated emission is more intensive than the absorption plus dissipation process ($A_3 - A_1 > 0$), we see from (32) that $D^2/N(t \to \infty) \to \infty$ whatever the initial size $N(0)$ of the photon population. Consequently, in this case the photon population is not stable at fluctuations.

A thorough analysis, which we do not present here, shows that the time variation of the photon population fluctuations towards the asymptotic limit (in the case when $A_3 - A_1 < 0$) is either monotonic or non-monotonic (with a maximum and/or a minimum), depending on the values of $N(0)$, A_1, A_2, A_3 and k.

We have considered the case when all the processses (the spontaneous and stimulated emission processes and the absorption plus dissipation processes) have comparable intensities. In the following we shall briefly analyse some extreme cases when one of these processes is negligible. First, we consider a very weak stimulated emission ($A_3 \simeq 0$). From (22) we see that the generating function is not defined in this case. New calculations give the following expression for $F(s, t)$:

$$F(s,t) = [1-(1-s)\exp(-A_1 t)]^{N(0)} - \exp\{(A_2/A_1)(1-s)[\exp(-A_1 t)-1]\}. \tag{33}$$

The new expressions for the mean photon number and variance are

$$\bar{N}(t) = N(0)[1-\exp(-A_1 t)] + A_2/A_1 \tag{34}$$

$$D^2 = \bar{N}(t) - N(0)\exp(-2A_1 t). \tag{35}$$

The photon population is globally stable, i.e. the value of $\bar{N}(t)$ is finite whatever the values of $N(0)$, A_1 and A_2. In the asymptotic limit $t \to \infty$ the mean photon number increases or decreases towards A_2/A_1. As we can see, the relaxation time A_1^{-1} is always finite. Also, $\bar{N}$ cannot diverge. Consequently, a phase transition is not possible in this case. From (35) we observe that in the limit $t \to \infty$ the distribution $P(N, t)$ is well approximated by a Poisson distribution. Consequently, the photon ensemble is characterized by negligible fluctuations compared with the mean photon number $\bar{N}$. The photon population is stable at fluctuations.

Second, we consider a very weak spontaneous emission and influence of the ambient ($A_2 \simeq 0$). This case was analysed also by Schöll and Landsberg (1983) by means of the rate equations method. New calculations performed by using the generating function (22) under the condition $A_2 = 0$ give the following expressions for the mean photon number and variance:

$$\bar{N}(t) = N(0)\exp[(A_2 - A_1)t] \tag{36}$$

$$D^2 = \bar{N}(t)[(A_1 + A_3)/(A_1 - A_3)]\{1 - \exp[(A_3 - A_1)t]\}. \tag{37}$$

When stimulated emission is more intensive than absorption plus dissipation ($A_3 - A_1 > 0$), both the mean photon number $\bar{N}(t)$ and the variance $D^2(t)$ increases in time indefinitely. Consequently, the photon population is unstable. When stimulated emission is weaker than absorption plus dissipation ($A_3 - A_1 < 0$), both $\bar{N}(t)$ and $D^2(t)$ decrease monotonically towards zero. The photon population disappears. Moreover, $N = 0$ is an absorbing state. Indeed, in the case of $A_2 = 0$ and $N(0) = 0$ the probability $P(0, t > 0)$ becomes unity (see (29)).

When the two processes have the same intensities ($A_3 = A_1$), the relaxation time $(A_3 - A_1)^{-1}$ becomes infinity. Also, the mean photon number $\bar{N}(t)$ has a constant value during time and the fluctuations are very large, because D^2 diverges. The phase transition analogy applies. Note that, when $A_3 = A_1$ and $t > 0$, the constant mean photon number $\bar{N}(t) = N(0)$ has a jump compared with the value $\bar{N}(t)$ corresponding to

$A_3 - A_1 < 0$. This is a characteristic of the first-order phase transition. In the same case that we studied ($A_2 = 0$), Schöll and Landsberg (1983) emphasized a second-order phase transition. However, in their model the phase transition occurs when the transition probabilities per unit time for non-radiative excitations and recombinations have the same non-zero value. This point is not accessible to our model which neglected the non-radiative processes.

4. Steady-state photon statistics

The steady state of the photon population may be obtained in the asymptotic limit $t \to \infty$. This limit is finite only when the absorption plus dissipation process is more intensive than stimulated emission ($A_3 - A_1 < 0$) (see section 3). With this in mind, equation (22) becomes

$$F(s, t \to \infty) = |(A_1 - A_3)/(A_1 - A_3 s)|^{A_2/A_3}. \tag{38}$$

The mean number of photons may be derived from (25):

$$\bar{N}_{ss} = A_2/(A_1 - A_3). \tag{39}$$

We intend to rewrite equation (39) in a more familiar form. First, we must remember that the electron states obey a quasi-Fermi distribution for which the chemical potential has been replaced by the quasichemical potential μ (Landsberg 1986). We define

$$\eta_G \equiv (E_J - E_J)/k_B T = h\nu/k_B T \qquad \tilde{\mu} \equiv (\mu_J - \mu_J)/k_B T \equiv \mu/k_B T \tag{40}$$

where k_B is Boltzmann's constant. By using (40) the following result was obtained (Landsberg 1986):

$$\beta/\alpha = \exp(\eta_G - \tilde{\mu}). \tag{41}$$

If we use equations (14)–(16) and (41) we can rewrite equation (39) in the form

$$\bar{N}_{ss} = 1/[f^{-1} \exp(\eta_G - \tilde{\mu}) - 1] \tag{42}$$

where

$$f = (B_e^{sp} + kN_0/\alpha)/[B_a + k(N_0 + 1)/\beta + (\alpha/\beta)(B_e^{sp} - B_e^{st})]. \tag{43}$$

Because $B_a = B_e^{sp} = B_e^{st}$ our equations (42) and (43) reduce to those presented by Schöll and Landsberg (1983) and Landsberg (1986). Moreover, if dissipation is negligible, then $f = 1$ and $\bar{N}_{ss}$ corresponds to a black-body distribution with an effective chemical potential $\tilde{\mu}$. Discussions on this last case can be found in the work of Landsberg (1981), Wurfel and Ruppel (1981), Wurfel (1982) and Landsberg (1986).

When the influence of the ambient is strong ($kN_0 \to \infty$) we see that

$$f \to [(N_0 + 1)/N_0](\alpha/\beta). \tag{44}$$

Consequently, from (42) we obtain $\bar{N}_{ss} \to N_0$. In other words, the photon population attained thermal equilibrium.

The steady-state value of the ratio $D(t)/\bar{N}(t)$ can be computed with (32) in the limit $t \to \infty$:

$$(D/\bar{N})_{ss} = (1/\bar{N}_{ss} + A_3/A_2)^{1/2}. \tag{45}$$

From (45) we see that generally the steady-state photon distribution $P(N)$ is not

Poissonian. Consequently, the photon population has larger fluctuations. The fluctuations become negligible when stimulated emission decreases ($A_3 \to 0$).

We obtain new information if we compute the distribution $P(N)$ by means of (38) and (23):

$$P(N) = A(N)(A_3/A_1)^N \tag{46}$$

where

$$A(N) = (1/N!)(A_2/A_3)(A_2/A_3 + 1)(A_2/A_3 + 2)\ldots[A_2/A_3 + (N-1)] \times (1 - A_3/A_1)^{A_2/A_3}. \tag{47}$$

By using (14), (16), (46) and (47) we find that

$$P(N) = \tilde{A}(N) \exp[(\mu N - Nh\nu)/k_B T] \tag{48}$$

where

$$\tilde{A}(N) = A(N)(B_a/B_e^{st} + k/\beta B_e^{st})^{-N}. \tag{49}$$

Equation (48) will be used in order to determine certain steady-state properties of the photon population. First, let us compute the entropy of the photons from (Landsberg 1986)

$$S = -k_B \sum_{N=0}^{\infty} P(N) \ln[P(N)]. \tag{50}$$

From (48) and (50) we obtain

$$S = -k_B(\overline{\ln \tilde{A}}) + (\mu/T)\bar{N} - (1/T)\bar{E} \tag{51}$$

where

$$\bar{N} \equiv \sum_{N=0}^{\infty} NP(N) = \bar{N}_{ss} \tag{52}$$

$$\bar{E} \equiv \sum_{N=0}^{\infty} h\nu NP(N) = \bar{N}_{ss} h\nu \tag{53}$$

$$\overline{\ln \tilde{A}} \equiv \sum_{N=0}^{\infty} \ln[\tilde{A}(N)]P(N). \tag{54}$$

Now, we may use (51) and the Euler equation (Landau and Lifshitz 1967, p 187)

$$\bar{E} = TS - pV + \mu\bar{N} \tag{55}$$

where p and V are pressure and the volume of the photon ensemble. We obtain

$$pV = -k_B T \overline{\ln \tilde{A}}. \tag{56}$$

This is the equation of state of the photon population. Note that the equation of state is characterized by negligible fluctuations only when $P(N)$ approaches a Poisson-type distribution, i.e. when the stimulated emission becomes negligible ($A_3 \to 0$) (see the end of section 3).

Let us briefly analyse the case when the dissipation constant is weak ($k \to 0$). By taking into account that $B_e^{st} = B_e^{sp} = B_a$, from (15) and (16) we obtain $A_2 = A_3$. Then (47) gives

$$A(N) = 1 - A_3/A_1 \tag{57}$$

and from (15), (16), (41) and (48) we obtain

$$P(N) = [1 - \exp(\tilde{\mu} - \eta_G)] \exp[(\mu N - Nh\nu)/k_B T]. \tag{58}$$

In this case we use $f \to 1$ (see (43)). Then, from (42) and (58) and using the normalization condition we find that

$$P(N) = [1/(1 + \bar{N}_{ss})][\bar{N}_{ss}/(1 + \bar{N}_{ss})]^N. \tag{59}$$

As we see, $P(N)$ is not a Poisson distribution. Another form of $P(N)$ may be obtained using (54), (56) and (57)

$$P(N) = \exp[(-pV + \mu N - Nh\nu)/k_B T]. \tag{60}$$

The equation of state (56) has a simple form if we use (48), (54), (56) and $f = 1$:

$$pV = k_B T \ln(1 + \bar{N}_{ss}). \tag{61}$$

However, note that this equation describes a system with important fluctuations, because $P(N)$ is not Poissonian. In the case of a dilute system, $0 < \bar{N}_{ss} \leqslant 1$. Then, we can develop $\ln(1 + \bar{N}_{ss})$ and we obtain

$$pV = k_B \bar{N}_{ss} T(1 - \bar{N}_{ss}/2 + \bar{N}_{ss}^2/3 - \ldots). \tag{62}$$

Landau and Lifshitz (1967, p 188) show a nearly similar result in the case of a degenerate Bose gas in equilibrium. Note that $\bar{N}_{ss} \ll 1$ implies in our case $\eta_G \gg \tilde{\mu}$.

5. Conclusions

In this paper we studied a system of coupled photons and electrons by taking into account the radiative transitions and photon dissipation. The main hypothesis that we accepted was the slaving principle for adiabatic elimination of electrons. This assumption is justified because the lifetime of photons is usually much greater than the lifetime of electrons. Consequently, the electrons obey instantaneously the 'orders' of the photons and the electron number n can be eliminated without increasing the degree of the time derivatives. The slaving principle allowed us to obtain a master equation dependent only on the photon number N. However, this equation depends indirectly on the electronic component of the system, by means of three parameters, namely A_1, A_2 and A_3, in equations (14)–(16).

The main conclusions that we obtained are the following.

(i) The photon population is stable when the absorption plus dissipation process is more intensive than stimulated emission. The two kinds of processes have equal rates at the laser threshold. In this case the photon relaxation time becomes infinite and a phase transition analogy holds.

(ii) Fluctuations may cause the photon number to become zero. This is possible for both stable and unstable populations. A stable population may definitely disappear only when spontaneous emission is zero (i.e. when $k = 0$ and $n = N_e - N_f$ (see equation (15))).

(iii) Generally, the fluctuations are not negligible compared with the mean photon number. When stimulated emission is zero, the photon ensemble is globally stable and the fluctuations are negligible. When the spontaneous emission and the influence of the ambient are zero, the photon population is either stable or unstable, depending on the sign of $A_3 - A_1$. In this particular case a first-order phase transition was explicitly established at the laser threshold ($A_3 = A_1$).

(iv) We proved that the steady-state photon distribution is not Poissonian and is characterized by large fluctuations. The fluctuations become negligible only when the spontaneous emission and the influence of the ambient are much stronger than stimulated emission.

(v) When dissipation is zero, an effective chemical potential of the photons was emphasized.

(vi) The photon equation of state is given by (56). When dissipation is weak and dilution increases, the photon population has an equation of state similar to that of a degenerate Bose gas in equilibrium.

Appendix

The characteristic system attached to equation (23) is

$$dt/1 = ds/(1-s)(sA_3 - A_1) = dF/A_2(s-1)F. \tag{A1}$$

By integrating equations (A1) we obtain

$$|(sA_3 - A_1)/(s-1)| \exp[-(A_3 - A_1)t] = C_1 \tag{A2}$$

$$|F(sA_3 - A_1)|^{A_2/A_3} = C_2 \tag{A3}$$

where C_1 and C_2 are constants. Equation (23) has the solution

$$W(C_1, C_2) = 0 \tag{A4}$$

or, by using (A2) and (A3):

$$F = |sA_3 - A_1|^{-A_2/A_3} g(|(sA_3 - A_1)/(s-1)| \exp[-(A_3 - A_1)t]) \tag{A5}$$

where W and g are arbitrary functions.

The form of g may be determined by using equation (26):

$$F(s, 0) = s^{N(0)}. \tag{A6}$$

Then (A5) becomes

$$s^{N(0)} = |sA_3 - A_1|^{-A_2/A_3} g(|(sA_3 - A_1)/(s-1)|). \tag{A7}$$

We define

$$y = s^{N(0)} |sA_3 - A_1|^{A_2/A_3}. \tag{A8}$$

Then, from equation (A7) we obtain

$$x = g^{-1}(y) = |(sA_3 - A_1)/(s-1)| \tag{A9}$$

where g^{-1} is the inverse of the function g. From (A9) we may write

$$s = (A_1 - g^{-1})/(A_3 - g^{-1}). \tag{A10}$$

Consequently

$$s^{N(0)} = [(A_1 - g^{-1})/(A_3 - g^{-1})]^{N(0)}. \tag{A11}$$

By using again (A10) we find that

$$sA_3 - A_1 = (A_1 - A_3)g^{-1}/(A_3 - g^{-1}). \tag{A12}$$

From (A8), (A11) and (A12) we obtain

$$y = [(A_1 - g^{-1})/(A_3 - g^{-1})]^{N(0)}[(A_1 - A_3)g^{-1}/(A_3 - g^{-1})]^{A_2/A_3} \tag{A13}$$

but (A9) shows that

$$y = g(x) \qquad g^{-1}(y) = x. \tag{A14}$$

Consequently, (A13) becomes

$$g(x) = [(A_1 - x)/(A_3 - x)]^{N(0)}[(A_1 - A_3)x/(A_3 - x)]^{A_2/A_3}. \tag{A15}$$

By using (A5) and (A15) we obtain equation (22).

References

Bharucha-Reid A T 1960 *Elements of the Theory of Markov Processes and Their Applications* (New York: McGraw-Hill)

Chel'tzov V F 1971 *Solid State Commun.* **9** 1343–6

DeGiorgio V and Scully M O 1970 *Phys. Rev.* A **2** 1170–7

Graham R and Haken H 1970 *Z. Phys.* **237** 31–9

Haken H 1970 *Encyclopedia of Physics* vol XXV/2c, ed S Flügge (Berlin: Springer) pp 249–71

—— 1975 *Rev. Mod. Phys.* **47** 67–121

Haught A F 1984 *ASME J. Solar Energy Eng.* **106** 3–15

Henry C H 1980 *J. Appl. Phys.* **51** 4494

Landau L and Lifshitz E 1967 *Physique Statistique* (Moscow: Mir)

Landsberg P T 1981 *J. Phys. C: Solid State Phys.* **14** L1025–7

—— 1986 *An Introduction to Nonequilibrium Problems Involving Electromagnetic Radiation, in Recent Development in Nonequilibrium Thermodynamics: Fluids and Related Topics* ed J Casas-Vazquez, D Jon and J M Rubi (Berlin: Springer) pp 224–67

Nicolis G and Prigogine I 1977 *Self-Organization in Nonequilibrium Systems* (New York: Wiley)

Ruppel W and Wurfel P 1980 *IEEE Trans. Electron. Devices* **ED-27** 877–82

Schöll E and Landsberg P T 1983 *J. Opt. Soc. Am.* **73** 1197–206

Wurfel P 1982 *J. Phys. C: Solid State Phys.* **15** 3967–85

Wurfel P and Ruppel W 1980 *Proc. 3rd Commission of the European Communities Conf. on Photovoltaic Solar Energy (Cannes, October 1980)* (Dordrecht: Reidel)

—— 1981 *J. Lumin.* **24–5** 925–8

—— 1985 *J. Phys. C: Solid State Phys.* **18** 2987